Laws of thought

Law of identity: Whatever is, is.

Law of contradiction: Nothing can both be and not be.

Law of excluded middle: Everything must either be or not be.

Primitive ideas

Elementary proposition (p, q, r, s)

Elementary propositional function

Assertion of elementary proposition (⊦)

Assertion of elementary propositional function

Negation (∼p)

Disjunction (p ∨ q)

Composing propositions

E.g.:

∼∼p

∼p ∨ q

p ∨ ∼q

∼p ∨ ∼q

∼(p ∨ q)

∼(∼p ∨ q)

∼(p ∨ ∼q)

∼(∼p ∨ ∼q)

r ∨ (p ∨ q)

(p ∨ q) ∨ r

∼(p ∨ q) ∨ r

(p ∨ q) ∨ (p ∨ r)

∼((p ∨ q) ∨ r)

∼(∼(p ∨ q) ∨ r)

$\sim((p \vee q) \vee (p \vee r))$

… etc.

Substituting

E.g.:

$p \vee q\ [p \coloneqq \sim p]$ is $\sim p \vee q$

$p \vee q\ [q \coloneqq p \vee r]$ is $p \vee (p \vee r)$

$p \vee q\ [p \coloneqq p \vee q,\ q \coloneqq p \vee r]$ is $(p \vee q) \vee (p \vee r)$

$(p \vee q) \vee (p \vee r)\ [p \coloneqq \sim q,\ q \coloneqq p \vee r]$ is $(\sim q \vee (p \vee r)) \vee (\sim q \vee r)$

… etc.

Definition of implication

*1.01 $p \to q$ is an abbreviation for $\sim p \vee q$

E.g.:

$(p \vee p) \to p$ is an abbreviation for $\sim(p \vee p) \vee p$

$\sim((p \to q) \to r)$ is an abbreviation for $\sim(\sim(\sim p \vee q) \vee r)$

… etc.

Primitive propositions

*1.1 Anything implied by a true elementary proposition is true. (*Rule of inference*)

*1.2 $\vdash (p \vee p) \to p$ (*Principle of tautology*)

*1.3 $\vdash q \to (p \vee q)$ (*Principle of addition*)

*1.4 $\vdash (p \vee q) \to (q \vee p)$ (*Principle of permutation*)

*1.5 $\vdash (p \vee (q \vee r)) \to (q \vee (p \vee r))$ (*Associative principle*)

*1.6 $\vdash (q \to r) \to ((p \vee q) \to (p \vee r))$ (*Principle of summation*)

*1.7 Negation of an elementary proposition is an elementary proposition.

*1.71 Disjunction of two elementary propositions is an elementary proposition.

*1.72 In a disjunction of two elementary propositional functions containing the same variables we can identify these variables.

Deduced propositions

*2.01 $\vdash (p \to \sim p) \to \sim p$ (*Principle of the reductio ad absurdum*)

Proof:

*1.2 [p ≔ ~p]	$\vdash (\sim p \lor \sim p) \to \sim p$	(1)
(1), *1.01	$\vdash (p \to \sim p) \to \sim p$	

*2.02 $\vdash q \to (p \to q)$ (*Principle of simplification*)

Proof:

*1.3 [p ≔ ~p]	$\vdash q \to (\sim p \lor q)$	(1)
(1), *1.01	$\vdash q \to (p \to q)$	

*2.03 $\vdash (p \to \sim q) \to (q \to \sim p)$ (*Principle of transposition*)

Proof:

*1.4 [p ≔ ~p, q ≔ ~q]	$\vdash (\sim p \lor \sim q) \to (\sim q \lor \sim p)$	(1)
(1), *1.01	$\vdash (p \to \sim q) \to (q \to \sim p)$	

*2.04 $\vdash (p \to (q \to r)) \to (q \to (p \to r))$ (*Commutative principle*)

Proof:

*1.5 [p ≔ ~p, q ≔ ~q]	$\vdash (\sim p \lor (\sim q \lor r)) \to (\sim q \lor (\sim p \lor r))$	(1)
(1), *1.01	$\vdash (p \to (q \to r)) \to (q \to (p \to r))$	

*2.05 $\vdash (q \to r) \to ((p \to q) \to (p \to r))$ (*Principle of the syllogism*)

Proof:

*1.6 [p ≔ ~p]	$\vdash (q \to r) \to ((\sim p \lor q) \to (\sim p \lor r))$	(1)
(1), *1.01	$\vdash (q \to r) \to ((p \to q) \to (p \to r))$	

*2.06 $\vdash (p \to q) \to ((q \to r) \to (p \to r))$ (*Principle of the syllogism*)

Proof:

*2.04 [p ≔ q → r, q ≔ p → q, r ≔ p → r]

	$\vdash ((q \to r) \to ((p \to q) \to (p \to r)))$	
	$\to ((p \to q) \to ((q \to r) \to (p \to r)))$	(1)
*2.05	$\vdash (q \to r) \to ((p \to q) \to (p \to r))$	(2)
(1), (2), *1.1	$\vdash (p \to q) \to ((q \to r) \to (p \to r))$	

*2.07 $\vdash p \to (p \lor p)$

Proof:

*1.3 [q ≔ p] $\vdash p \to (p \lor p)$

*2.08 $\vdash p \to p$ (*Principle of identity*)

Proof:

*2.05 [q ≔ p ∨ p, r ≔ p]

$\vdash ((p \lor p) \to p) \to ((p \to (p \lor p)) \to (p \to p))$ (1)

*1.2 $\vdash (p \lor p) \to p$ (2)

(1), (2), *1.1 $\vdash (p \to (p \lor p)) \to (p \to p)$ (3)

*2.07 $\vdash p \to (p \lor p)$ (4)

(3), (4), *1.1 $\vdash p \to p$

*2.1 $\vdash \sim p \lor p$ (*Law of excluded middle*)

Proof:

*2.08, *1.01 $\vdash \sim p \lor p$

*2.11 $\vdash p \lor \sim p$ (*Law of excluded middle*)

Proof:

*1.4 [p ≔ ~p, q ≔ p] $\vdash (\sim p \lor p) \to (p \lor \sim p)$ (1)

*2.1 $\vdash \sim p \lor p$ (2)

(1), (2), *1.1 $\vdash p \lor \sim p$

*2.12 $\vdash p \to \sim\sim p$ (*Principle of double negation*)

Proof:

*2.11 [p ≔ ~p] $\vdash \sim p \lor \sim\sim p$ (1)

(1), *1.01 $\vdash p \to \sim\sim p$

*2.13 $\vdash p \lor \sim\sim\sim p$

Proof:

*1.6 [q ≔ ~p, r ≔ ~~~p]

$\vdash (\sim p \to \sim\sim\sim p) \to ((p \lor \sim p) \to (p \lor \sim\sim\sim p))$ (1)

*2.12 [p ≔ ~p] $\vdash \sim p \to \sim\sim\sim p$ (2)

(1), (2), *1.1 ⊦ (p ∨ ~p) → (p ∨ ~~~p) (3)

*2.11 ⊦ p ∨ ~p (4)

(3), (4), *1.1 ⊦ p ∨ ~~~p

*2.14 ⊦ ~~p → p (*Principle of double negation*)

Proof:

*1.4 [q ≔ ~~~p] ⊦ (p ∨ ~~~p) → (~~~p ∨ p) (1)

*2.13 ⊦ p ∨ ~~~p (2)

(1), (2), *1.1 ⊦ ~~~p ∨ p (3)

(3), *1.01 ⊦ ~~p → p

*2.15 ⊦ (~p → q) → (~q → p) (*Principle of transposition*)

*2.16 ⊦ (p → q) → (~q → ~p) (*Principle of transposition*)

*2.17 ⊦ (~q → ~p) → (p → q) (*Principle of transposition*)

*2.2 ⊦ p → (p ∨ q)

Proof:

*1.3 [p ≔ q, q ≔ p] ⊦ p → (q ∨ p) (1)

*1.4 [p ≔ q, q ≔ p] ⊦ (q ∨ p) → (p ∨ q) (2)

*2.06 [q ≔ q ∨ p, r ≔ p ∨ q]

⊦ (p → (q ∨ p)) → (((q ∨ p) → (p ∨ q)) → (p → (p ∨ q))) (3)

(1), (3), *1.11 ⊦ ((q ∨ p) → (p ∨ q)) → (p → (p ∨ q)) (4)

(2), (4), *1.11 ⊦ p → (p ∨ q)

We will use an abbreviated form of this proof:

*1.3 [p ≔ q, q ≔ p] ⊦ p → (q ∨ p) (1)

*1.4 [p ≔ q, q ≔ p] ⊦ (q ∨ p) → (p ∨ q) (2)

(1), (2), *2.06 ⊦ p → (p ∨ q)

*2.21 ⊦ ~p → (p → q)

Proof:

*2.2 [p ≔ ~p] ⊦ ~p → (~p ∨ q) (1)

(1), *1.01 ⊦ ~p → (p → q)

*2.3 $\vdash (p \lor (q \lor r)) \to (p \lor (r \lor q))$

Proof:

*1.4 [p ≔ q, q ≔ r] $\vdash (q \lor r) \to (r \lor q)$ (1)

*1.6 [q ≔ q ∨ r, r ≔ r ∨ q] $\vdash ((q \lor r) \to (r \lor q)) \to ((p \lor (q \lor r)) \to (p \lor (r \lor q)))$ (2)

(1), (2), *1.11 $\vdash (p \lor (q \lor r)) \to (p \lor (r \lor q))$

*2.31 $\vdash (p \lor (q \lor r)) \to ((p \lor q) \lor r)$

Proof:

*2.3 $\vdash (p \lor (q \lor r)) \to (p \lor (r \lor q))$ (1)

*1.5 [q ≔ r, r ≔ q] $\vdash (p \lor (r \lor q)) \to (r \lor (p \lor q))$ (2)

(1), (2), *2.05 $\vdash (p \lor (q \lor r)) \to (r \lor (p \lor q))$ (3)

*1.4 [p ≔ r, q ≔ p ∨ q] $\vdash (r \lor (p \lor q)) \to ((p \lor q) \lor r)$ (4)

(3), (4), *2.05 $\vdash (p \lor (q \lor r)) \to ((p \lor q) \lor r)$

We will use an abbreviated form of this proof:

*2.3 $\vdash (p \lor (q \lor r)) \to (p \lor (r \lor q))$ (1)

*1.5 [q ≔ r, r ≔ q] $\vdash \ldots \to (r \lor (p \lor q))$ (2)

*1.4 [p ≔ r, q ≔ p ∨ q] $\vdash \ldots \to ((p \lor q) \lor r)$

*2.32 $\vdash ((p \lor q) \lor r) \to (p \lor (q \lor r))$

Proof:

*1.4 [p ≔ p ∨ q, q ≔ r] $\vdash ((p \lor q) \lor r) \to (r \lor (p \lor q))$ (1)

*1.5 [p ≔ r, q ≔ p, r ≔ q] $\vdash \ldots \to (p \lor (r \lor q))$ (2)

*2.3 [q ≔ r, r ≔ q] $\vdash \ldots \to (p \lor (q \lor r))$

*2.38 $\vdash (q \to r) \to ((q \lor p) \to (r \lor p))$

Proof:

*2.06 [p ≔ q ∨ p, q ≔ p ∨ q, r ≔ p ∨ r]

$\vdash ((q \lor p) \to (p \lor q))$

$\to (((p \lor q) \to (p \lor r)) \to ((q \lor p) \to (p \lor r)))$ (1)

*1.4 [p ≔ q, q ≔ p] $\vdash (q \lor p) \to (p \lor q)$ (2)

(1), (2), *1.11 $\vdash ((p \lor q) \to (p \lor r)) \to ((q \lor p) \to (p \lor r))$ (3)

*2.05 [p ≔ q ∨ p, q ≔ p ∨ r, r ≔ r ∨ p]

$\vdash ((p \lor r) \to (r \lor p))$

	→ (((q ∨ p) → (p ∨ r)) → ((q ∨ p) → (r ∨ p)))	(4)
*1.4 [q ≔ r]	⊦ (p ∨ r) → (r ∨ p)	(5)
(4), (5), *1.11	⊦ ((q ∨ p) → (p ∨ r)) → ((q ∨ p) → (r ∨ p))	(6)
(3), (6), *2.06	⊦ ((p ∨ q) → (p ∨ r)) → ((q ∨ p) → (r ∨ p))	(7)
*1.6	⊦ (q → r) → ((p ∨ q) → (p ∨ r))	(8)
(7), (8), *2.05	⊦ (q → r) → ((q ∨ p) → (r ∨ p))	

*2.53 ⊦ (p ∨ q) → (∼p → q)

Proof:

*2.12	⊦ p → ∼∼p	(1)
*2.38 [p ≔ q, q ≔ p, r ≔ ∼∼p]		
	⊦ (p → ∼∼p) → ((p ∨ q) → (∼∼p ∨ q))	(2)
(1), (2), *1.11	⊦ (p ∨ q) → (∼∼p ∨ q)	(3)
(3), *1.01	⊦ (p ∨ r) → (∼p → q)	

Definition of conjunction

*3.01 p ∧ q is an abbreviation for ∼(∼p ∨ ∼q)

E.g.:

p ∧ ∼q is an abbreviation for ∼(∼p ∨ ∼∼q)

(p ∧ q) → (q ∧ p) is an abbreviation for ∼∼(∼p ∨ ∼q) ∨ ∼(∼q ∨ ∼p)

… etc.

Deduced propositions

*3.1 ⊦ (p ∧ q) → ∼(∼p ∨ ∼q)

Proof:

*2.08 [p ≔ ∼(∼p ∨ ∼q)]	⊦ ∼(∼p ∨ ∼q) → ∼(∼p ∨ ∼q)	(1)
(1), *3.01	⊦ (p ∧ q) → ∼(∼p ∨ ∼q)	

We will use an abbreviated form of this proof:

*2.08 [p ≔ ∼(∼p ∨ ∼q)], *3.01   ⊦ (p ∧ q) → ∼(∼p ∨ ∼q)

*3.11 $\vdash \sim(\sim p \lor \sim q) \to (p \land q)$

Proof:

*2.08 [$p \coloneqq \sim(\sim p \lor \sim q)$], *3.01 $\vdash \sim(\sim p \lor \sim q) \to (p \land q)$

*3.12 $\vdash \sim p \lor (\sim q \lor (p \land q))$

Proof:

*2.11 [$p \coloneqq \sim p \lor \sim q$], *3.01	$\vdash (\sim p \lor \sim q) \lor (p \land q)$	(1)
*2.32 [$p \coloneqq \sim p, q \coloneqq \sim q, r \coloneqq p \land q$]	$\vdash ((\sim p \lor \sim q) \lor (p \land q)) \to (\sim p \lor (\sim q \lor (p \land q)))$	(2)
(1), (2), *1.11	$\vdash \sim p \lor (\sim q \lor (p \land q))$	

*3.13 $\vdash \sim(p \land q) \to (\sim p \lor \sim q)$

Proof:

*3.11	$\vdash \sim(\sim p \lor \sim q) \to (p \land q)$	(1)
*2.15 [$p \coloneqq \sim p \lor \sim q, q \coloneqq p \land q$]	$\vdash (\sim(\sim p \lor \sim q) \to (p \land q)) \to (\sim(p \land q) \to (\sim p \lor \sim q))$	(2)
(1), (2), *1.11	$\vdash \sim(p \land q) \to (\sim p \lor \sim q)$	

We will use an abbreviated form of this proof:

*3.11	$\vdash \sim(\sim p \lor \sim q) \to (p \land q)$	(1)
(1), *2.15	$\vdash \sim(p \land q) \to (\sim p \lor \sim q)$	

*3.14 $\vdash (\sim p \lor \sim q) \to \sim(p \land q)$

Proof:

*3.1	$\vdash (p \land q) \to \sim(\sim p \lor \sim q)$	(1)
(1), *2.03	$\vdash (\sim p \lor \sim q) \to \sim(p \land q)$	

*3.2 $\vdash p \to (q \to (p \land q))$

Proof:

*3.12, *1.01 $\vdash p \to (q \to (p \land q))$

*3.22 $\vdash (p \land q) \to (q \land p)$ (*Commutative law for logical multiplication*)

Proof:

*3.13 [$p \coloneqq q, q \coloneqq p$]	$\vdash \sim(q \land p) \to (\sim q \lor \sim p)$	(1)

*1.4 $[p \coloneqq \sim q, q \coloneqq \sim p]$ $\vdash \ldots \rightarrow (\sim p \lor \sim q)$ (2)

*3.14 $\vdash \ldots \rightarrow \sim(p \land q)$ (3)

(3), *2.17 $\vdash (p \land q) \rightarrow (q \land p)$

*3.24 $\vdash \sim(p \land \sim p)$ (*Law of contradiction*)

Proof:

*2.11 $[p \coloneqq \sim p]$ $\vdash \sim p \lor \sim\sim p$ (1)

*3.14 $[q \coloneqq \sim p]$ $\vdash (\sim p \lor \sim\sim p) \rightarrow \sim(p \land \sim p)$ (2)

(1), (2), *1.11 $\vdash \sim(p \land \sim p)$

*3.26 $\vdash (p \land q) \rightarrow p$ (*Principle of simplification*)

Proof:

*2.02 $[p \coloneqq q, q \coloneqq p]$, *1.01 $\vdash \sim p \lor (\sim q \lor p)$ (1)

*2.31 $[p \coloneqq \sim p, q \coloneqq \sim q, r \coloneqq p]$ $\vdash (\sim p \lor (\sim q \lor p)) \rightarrow ((\sim p \lor \sim q) \lor p)$ (2)

(1), (2), *1.11 $\vdash (\sim p \lor \sim q) \lor p$ (3)

*2.53 $[p \coloneqq \sim p \lor \sim q, q \coloneqq p]$ $\vdash ((\sim p \lor \sim q) \lor p) \rightarrow (\sim(\sim p \lor \sim q) \rightarrow p)$ (4)

(3), (4), *1.11 $\vdash \sim(\sim p \lor \sim q) \rightarrow p$ (5)

(5), *3.01 $\vdash (p \land q) \rightarrow p$

*3.27 $\vdash (p \land q) \rightarrow q$ (*Principle of simplification*)

Proof:

*3.22 $\vdash (p \land q) \rightarrow (q \land p)$ (1)

*3.26 $[p \coloneqq q, q \coloneqq p]$ $\vdash \ldots \rightarrow q$

*3.3 $\vdash ((p \land q) \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$ (*Principle of exportation*)

Proof:

*2.08 $[p \coloneqq (p \land q) \rightarrow r]$, *3.01 $\vdash ((p \land q) \rightarrow r) \rightarrow (\sim(\sim p \lor \sim q) \rightarrow r)$ (1)

*2.15 $[p \coloneqq \sim p \lor \sim q, q \coloneqq r]$ $\vdash \ldots \rightarrow (\sim r \rightarrow (\sim p \lor \sim q))$ (2)

*2.08 $[p \coloneqq \sim r \rightarrow (\sim p \lor \sim q)]$, *1.01 $\vdash \ldots \rightarrow (\sim r \rightarrow (p \rightarrow \sim q))$ (3)

*2.04 $[p \coloneqq \sim r, q \coloneqq p, r \coloneqq \sim q]$ $\vdash \ldots \rightarrow (p \rightarrow (\sim r \rightarrow \sim q))$ (4)

*2.17 $[p \coloneqq r]$ $\vdash (\sim r \rightarrow \sim q) \rightarrow (q \rightarrow r)$ (5)

*2.05 $[q \coloneqq \sim r \rightarrow \sim q, r \coloneqq q \rightarrow r]$ $\vdash ((\sim r \rightarrow \sim q) \rightarrow (q \rightarrow r))$
$\rightarrow ((p \rightarrow (\sim r \rightarrow \sim q)) \rightarrow (p \rightarrow (q \rightarrow r)))$ (6)

(5), (6), *1.11 $\vdash (p \to (\sim r \to \sim q)) \to (p \to (q \to r))$ (7)

(4), (7), *2.06 $\vdash ((p \land q) \to r) \to (p \to (q \to r))$

*3.31 $\vdash (p \to (q \to r)) \to ((p \land q) \to r)$ (*Principle of importation*)

Proof:

*2.08 $[p := p \to (q \to r)]$, *1.01 $\vdash (p \to (q \to r)) \to (\sim p \lor (\sim q \lor r))$ (1)

*2.31 $[p := \sim p, q := \sim q]$ $\vdash \ldots \to ((\sim p \lor \sim q) \lor r)$ (2)

*2.53 $[p := \sim p \lor \sim q, q := r]$ $\vdash \ldots \to (\sim(\sim p \lor \sim q) \to r)$ (3)

*2.08 $[p := \sim(\sim p \lor \sim q) \to r]$, *3.01 $\vdash \ldots \to ((p \land q) \to r)$

*3.33 $\vdash ((p \to q) \land (q \to r)) \to (p \to r)$ (*Principle of the syllogism*)

*3.34 $\vdash ((q \to r) \land (p \to q)) \to (p \to r)$ (*Principle of the syllogism*)

*3.35 $\vdash (p \land (p \to q)) \to q$ (*Principle of assertion*)

*3.43 $\vdash ((p \to q) \land (p \to r)) \to (p \to (q \land r))$ (*Principle of composition*)

*3.45 $\vdash (p \to q) \to ((p \land r) \to (q \land r))$ (*Principle of the factor*)

*3.47 $\vdash ((p \to r) \land (q \to s)) \to ((p \land q) \to (r \land s))$

Definition of equivalence

*4.01 $p \leftrightarrow q$ is an abbreviation for $(p \to q) \land (q \to p)$

E.g.:

$(p \land q) \leftrightarrow (q \land p)$ is an abbreviation for

$((p \land q) \to (q \land p)) \land ((q \land p) \to (p \land q))$

or for $(\sim(p \land q) \lor (q \land p)) \land (\sim(q \land p) \lor (p \land q))$

or for $\sim(\sim(\sim(p \land q) \lor (q \land p)) \lor \sim(\sim(q \land p) \lor (p \land q)))$

or for $\sim(\sim(\sim\sim(\sim p \lor \sim q) \lor \sim(\sim q \lor \sim p)) \lor \sim(\sim\sim(\sim q \lor \sim p) \lor \sim(\sim p \lor \sim q)))$

… etc.

Deduced propositions

*4.1 $\vdash (p \to q) \leftrightarrow (\sim q \to \sim p)$ (*Principle of transposition*)

*4.11 $\vdash (p \leftrightarrow q) \leftrightarrow (\sim p \leftrightarrow \sim q)$ (*Principle of transposition*)

*4.13 $\vdash p \leftrightarrow \sim\sim p$ (*Principle of double negation*)

*4.2 $\vdash p \leftrightarrow p$ (*Principle of identity*)

*4.21 $\vdash (p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$

*4.22 $\vdash ((p \leftrightarrow q) \land (q \leftrightarrow r)) \rightarrow (p \leftrightarrow r)$

*4.24 $\vdash p \leftrightarrow (p \land p)$

*4.25 $\vdash p \leftrightarrow (p \lor p)$

*4.3 $\vdash (p \land q) \leftrightarrow (q \land p)$

*4.31 $\vdash (p \lor q) \leftrightarrow (q \lor p)$

*4.32 $\vdash ((p \land q) \land r) \leftrightarrow (p \land (q \land r))$

*4.33 $\vdash ((p \lor q) \lor r) \leftrightarrow (p \lor (q \lor r))$

*4.4 $\vdash (p \land (q \lor r)) \leftrightarrow ((p \land q) \lor (p \land r))$

*4.41 $\vdash (p \lor (q \land r)) \leftrightarrow ((p \lor q) \land (p \lor r))$

Deductions with *1.7

*2.01 $\vdash (p \rightarrow \sim p) \rightarrow \sim p$

Proof:

*1.2	$\vdash (p \lor p) \rightarrow p$	(1)
*1.7	$\sim p$ is an elementary propositional function	(2)
(2), *1.71,*1.72	$\sim p \lor \sim p$ is an elementary propositional function	(3)
(3), *1.7	$\sim(\sim p \lor \sim p)$ is an elementary propositional function	(4)
(2), (4), *1.71,*1.72	$\sim(\sim p \lor \sim p) \lor \sim p$ is an elementary propositional function	(5)
(5), *1.01	$(\sim p \lor \sim p) \rightarrow \sim p$ is an elementary propositional function	(6)
(6), (1) $[p \coloneqq \sim p]$	$\vdash (\sim p \lor \sim p) \rightarrow \sim p$	(7)
(7), *1.01	$\vdash (p \rightarrow \sim p) \rightarrow \sim p$	

... etc.

Primitive ideas – second step

Proposition (p, q, r, s)

Propositional function (φ, ψ, χ… with variables φx, φxy)

Assertion of proposition ($\vdash$)

Assertion of propositional function

Negation ($\sim p$)

Disjunction ($p \lor q$)

Universal quantifier ($\forall x\ \varphi x$)

Individual

Composing propositions – second step

E.g.:

$\sim \forall x\ \varphi x$

$\forall x \sim \varphi x$

$\sim \varphi y \lor \forall x\ \varphi x$

$\forall x\ (\varphi x \lor \psi x)$

$\forall x\ (\varphi x \lor p)$

$\forall x \sim \forall y\ \varphi xy$

$\forall x\ (\varphi x \lor \psi x) \lor \sim \forall x\ (\varphi x \lor \sim \psi x)$

$\sim (\forall x\ (\varphi x \lor \psi x) \lor \sim \forall x\ \varphi x)$

… etc.

Substituting – second step

E.g.:

$\forall x \sim \varphi x\ [\varphi x := \psi xy \lor \varphi x]$ is $\forall x \sim (\psi xy \lor \varphi x)$

$\sim \varphi y \lor \forall x\ \varphi x\ [\varphi x := \psi xy \lor \varphi x]$ is $\sim (\psi yy \lor \varphi y) \lor \forall x\ (\psi xy \lor \varphi x)$

$\sim \varphi y \lor \forall x\ \varphi x\ [\varphi x := \psi zx \lor \varphi z]$ is $\sim (\psi zy \lor \varphi z) \lor \forall x\ (\psi zx \lor \varphi z)$

$\sim \varphi y \lor \forall x\ \varphi x\ [y := p]$ is $\sim \varphi p \lor \forall x\ \varphi x$

$\sim \varphi y \lor \forall x\ \varphi x\ [\varphi x := \psi x, y := \chi]$ is $\sim \psi \chi \lor \forall x\ \psi x$

… etc.

Previous definitions – second step

Definitions of implication, conjunction, equivalence for non-elementary propositions

Definition of existential quantifier

*10.01 ∃x φx is an abbreviation for ~∀x ~φx

E.g.:

∃x ~φx is an abbreviation for ~∀x ~~φx

~∃x φx is an abbreviation for ~~∀x ~φx

~∃x ~φx is an abbreviation for ~~∀x ~~φx

∀x∃y ψxy is an abbreviation for ∀x~∀y ~ψxy

∀x (∃y ψxy ∨ ~∃y ψxy) is an abbreviation for ∀x (~∀y ~ψxy ∨ ~~∀y ~ψxy)

… etc.

Definition of being of the same type

*9.131 That two expressions are of the same type is an abbreviation for:

- both are individuals
- both are elementary propositions containing one individual
- both are elementary propositions containing two individuals
- … etc.
- both are elementary propositional functions containing one variable individual
- both are elementary propositional functions containing one variable individual and one constant individual
- both are elementary propositional functions containing two variable individuals
- ... etc.
- both are elementary propositions containing one expression of the same type
- ... etc.
- both are elementary propositional functions containing one variable expression of the same type
- … etc.
- both are propositions containing one bound variable individual
- ... etc.
- both are propositional functions containing one bound variable individual and one free variable individual
- ... etc.

- both are propositions containing one bound variable of the same type
- … etc.

Primitive propositions – second step

*1.2 to *1.4 for non-elementary propositions

*9.12 What is implied by a true premise is true. (*Rule of inference*)

*9.13 What is true of any, however chosen, is true of all. (*Rule of universal generalization*)

*9.14 If an expression containing a variable is significant, than it is significant with a constant in place of the variable if and only if they are of the same type. (*Rule of types*)

*9.15 There is a proposition containing a constant if and only if there is a propositional function with a variable in place of the constant.

*10.1 $\vdash \forall x\ \varphi x \rightarrow \varphi y$ (*Principle of deduction from the general to the particular*)

*10.12 $\vdash \forall x\ (p \lor \varphi x) \rightarrow (p \lor \forall x\ \varphi x)$

Deduced propositions – second step

*10.14 $\vdash (\forall x\ \varphi x \land \forall x\ \psi x) \rightarrow (\varphi y \land \psi y)$

Proof:

*10.1 $\vdash \forall x\ \varphi x \rightarrow \varphi y$ (1)

*10.1 [$\varphi x \coloneqq \psi x$] $\vdash \forall x\ \psi x \rightarrow \psi y$ (2)

*3.2 [$p \coloneqq \forall x\ \varphi x \rightarrow \varphi y,\ q \coloneqq \forall x\ \psi x \rightarrow \psi y$]

$\vdash (\forall x\ \varphi x \rightarrow \varphi y) \rightarrow ((\forall x\ \psi x \rightarrow \psi y)$

$\rightarrow ((\forall x\ \varphi x \rightarrow \varphi y) \land (\forall x\ \psi x \rightarrow \psi y)))$ (3)

(1), (3), *9.12 $\vdash (\forall x\ \psi x \rightarrow \psi y) \rightarrow ((\forall x\ \varphi x \rightarrow \varphi y) \land (\forall x\ \psi x \rightarrow \psi y))$ (4)

(2), (4), *9.12 $\vdash (\forall x\ \varphi x \rightarrow \varphi y) \land (\forall x\ \psi x \rightarrow \psi y)$ (5)

*3.47[$p \coloneqq \forall x\ \varphi x,\ q \coloneqq \forall x\ \psi x,\ r \coloneqq \varphi y,\ s \coloneqq \psi y$]

$\vdash ((\forall x\ \varphi x \rightarrow \varphi y) \land (\forall x\ \psi x \rightarrow \psi y))$

$\rightarrow ((\forall x\ \varphi x \land \forall x\ \psi x) \rightarrow (\varphi y \land \psi y))$ (6)

(5), (6), *9.12 $\vdash (\forall x\ \varphi x \land \forall x\ \psi x) \rightarrow (\varphi y \land \psi y)$

or an abbreviated form of this proof:

*10.1 $\vdash \forall x\ \varphi x \rightarrow \varphi y$ (1)

*10.1 [$\varphi x \coloneqq \psi x$] $\vdash \forall x\, \psi x \to \psi y$ (2)

(1), (2), *3.2 $\vdash (\forall x\, \varphi x \to \varphi y) \land (\forall x\, \psi x \to \psi y)$ (3)

(3), *3.47 $\vdash (\forall x\, \varphi x \land \forall x\, \psi x) \to (\varphi y \land \psi y)$

*10.2 $\vdash \forall x\, (p \lor \varphi x) \leftrightarrow (p \lor \forall x\, \varphi x)$

Proof:

*10.1 $\vdash \forall x\, \varphi x \to \varphi y$ (1)

(1), *1.6 $\vdash (p \lor \forall x\, \varphi x) \to (p \lor \varphi y)$ (2)

(2), *9.13 $\vdash \forall y\, ((p \lor \forall x\, \varphi x) \to (p \lor \varphi y))$ (3)

(3), *1.01 $\vdash \forall y\, (\sim(p \lor \forall x\, \varphi x) \lor (p \lor \varphi y))$ (4)

(4), *10.12 $\vdash \sim(p \lor \forall x\, \varphi x) \lor \forall y\, (p \lor \varphi y)$ (5)

(5), *1.01 $\vdash (p \lor \forall x\, \varphi x) \to \forall y\, (p \lor \varphi y)$ (6)

*10.12 $\vdash \forall x\, (p \lor \varphi x) \to (p \lor \forall x\, \varphi x)$ (7)

(6), (7), *3.2 $\vdash ((p \lor \forall x\, \varphi x) \to \forall y\, (p \lor \varphi y)) \land (\forall x\, (p \lor \varphi x) \to (p \lor \forall x\, \varphi x))$ (8)

(8), *4.01 $\vdash \forall x\, (p \lor \varphi x) \leftrightarrow (p \lor \forall x\, \varphi x)$

We will use an abbreviated form of this proof:

*10.1 $\vdash \forall x\, \varphi x \to \varphi y$ (1)

(1), *1.6 $\vdash (p \lor \forall x\, \varphi x) \to (p \lor \varphi y)$ (2)

(2), *9.13 $\vdash \forall y\, ((p \lor \forall x\, \varphi x) \to (p \lor \varphi y))$ (3)

(3), 10.12, *1.01 $\vdash (p \lor \forall x\, \varphi x) \to \forall y\, (p \lor \varphi y)$ (4)

*10.12 $\vdash \forall x\, (p \lor \varphi x) \to (p \lor \forall x\, \varphi x)$ (5)

(4), (5), *3.2, *4.01 $\vdash \forall x\, (p \lor \varphi x) \leftrightarrow (p \lor \forall x\, \varphi x)$

*10.21 $\vdash \forall x\, (p \to \varphi x) \leftrightarrow (p \to \forall x\, \varphi x)$

Proof:

*10.2 $\vdash \forall x\, (\sim p \lor \varphi x) \leftrightarrow (\sim p \lor \forall x\, \varphi x)$ (1)

(1), *1.01 $\vdash \forall x\, (p \to \varphi x) \leftrightarrow (p \to \forall x\, \varphi x)$

*10.22 $\vdash \forall x\, (\varphi x \land \psi x) \leftrightarrow (\forall x\, \varphi x \land \forall x\, \psi x)$

Proof:

*10.1 $\vdash \forall x\, (\varphi x \land \psi x) \to (\varphi y \land \psi y)$ (1)

*3.26 $\vdash \ldots \to \varphi y$ (2)

(2), *9.13 $\vdash \forall y\, (\forall x\, (\varphi x \land \psi x) \to \varphi y)$ (3)

*10.21	$\vdash \forall y\ (\forall x\ (\varphi x \wedge \psi x) \rightarrow \varphi y) \leftrightarrow (\forall x\ (\varphi x \wedge \psi x) \rightarrow \forall y\ \varphi y)$	(4)
(4), *4.01, *3.26	$\vdash \forall y\ (\forall x\ (\varphi x \wedge \psi x) \rightarrow \varphi y) \rightarrow (\forall x\ (\varphi x \wedge \psi x) \rightarrow \forall y\ \varphi y)$	(5)
(3), (5), *9.12	$\vdash \forall x\ (\varphi x \wedge \psi x) \rightarrow \forall y\ \varphi y$	(6)
by analogy, *3.27	$\vdash \forall x\ (\varphi x \wedge \psi x) \rightarrow \forall y\ \psi y$	(7)
(6), (7), *3.43	$\vdash \forall x\ (\varphi x \wedge \psi x) \rightarrow (\forall y\ \varphi y \wedge \forall y\ \psi y)$	(8)
*10.14	$\vdash (\forall x\ \varphi x \wedge \forall x\ \psi x) \rightarrow (\varphi y \wedge \psi y)$	(9)
(9), *9.13	$\vdash \forall y\ ((\forall x\ \varphi x \wedge \forall x\ \psi x) \rightarrow (\varphi y \wedge \psi y))$	(10)
*10.21	$\vdash \forall y\ ((\forall x\ \varphi x \wedge \forall x\ \psi x) \rightarrow (\varphi y \wedge \psi y))$	
	$\leftrightarrow ((\forall x\ \varphi x \wedge \forall x\ \psi x) \rightarrow \forall y\ (\varphi y \wedge \psi y))$	(11)
(11), *4.01, *3.26	$\vdash \forall y\ ((\forall x\ \varphi x \wedge \forall x\ \psi x) \rightarrow (\varphi y \wedge \psi y))$	
	$\rightarrow ((\forall x\ \varphi x \wedge \forall x\ \psi x) \rightarrow \forall y\ (\varphi y \wedge \psi y))$	(12)
(10), (12), *9.12	$\vdash (\forall x\ \varphi x \wedge \forall x\ \psi x) \rightarrow \forall y\ (\varphi y \wedge \psi y)$	(13)
(8), (13), *3.2, *4.01	$\vdash \forall x\ (\varphi x \wedge \psi x) \leftrightarrow (\forall x\ \varphi x \wedge \forall x\ \psi x)$	

We will use an abbreviated form of this proof:

*10.1	$\vdash \forall x\ (\varphi x \wedge \psi x) \rightarrow (\varphi y \wedge \psi y)$	(1)
*3.26	$\vdash \ldots \rightarrow \varphi y$	(2)
(2), *9.13, *10.21	$\vdash \forall x\ (\varphi x \wedge \psi x) \rightarrow \forall y\ \varphi y$	(3)
by analogy, *3.27	$\vdash \forall x\ (\varphi x \wedge \psi x) \rightarrow \forall y\ \psi y$	(4)
(3), (4), *3.43	$\vdash \forall x\ (\varphi x \wedge \psi x) \rightarrow (\forall y\ \varphi y \wedge \forall y\ \psi y)$	(5)
*10.14	$\vdash (\forall x\ \varphi x \wedge \forall x\ \psi x) \rightarrow (\varphi y \wedge \psi y)$	(6)
(6), *9.13, *10.21	$\vdash (\forall x\ \varphi x \wedge \forall x\ \psi x) \rightarrow \forall y\ (\varphi y \wedge \psi y)$	(7)
(5), (7), *3.2, *4.01	$\vdash \forall x\ (\varphi x \wedge \psi x) \leftrightarrow (\forall x\ \varphi x \wedge \forall x\ \psi x)$	

*10.24 $\vdash \varphi y \rightarrow \exists x\ \varphi x$

Proof:

*10.1	$\vdash \forall x \sim\varphi x \rightarrow \sim\varphi y$	(1)
(1), *2.03	$\vdash \varphi y \rightarrow \sim\forall x \sim\varphi x$	(2)
(2), *10.01	$\vdash \varphi y \rightarrow \exists x\ \varphi x$	

*10.25 $\vdash \forall x\ \varphi x \rightarrow \exists x\ \varphi x$

Proof:

*10.1	$\vdash \forall x\ \varphi x \rightarrow \varphi y$	(1)
*10.24	$\vdash \ldots \rightarrow \exists x\ \varphi x$	

*10.251 ⊦ $\forall x \sim \varphi x \rightarrow \sim\forall x\, \varphi x$

Proof:

*10.25	⊦ $\forall x\, \varphi x \rightarrow \exists x\, \varphi x$	(1)
(1), *10.01	⊦ $\forall x\, \varphi x \rightarrow \sim\forall x \sim\varphi x$	(2)
(2), *2.03	⊦ $\forall x \sim\varphi x \rightarrow \sim\forall x\, \varphi x$	

*10.252 ⊦ $\sim\exists x\, \varphi x \leftrightarrow \forall x \sim\varphi x$

Proof:

*2.08, *10.01	⊦ $\sim\exists x\, \varphi x \rightarrow \sim\sim\forall x \sim\varphi x$	(1)
*2.14	⊦ … $\rightarrow \forall x \sim\varphi x$	(2)
*2.08, *10.01	⊦ $\exists x\, \varphi x \rightarrow \sim\forall x \sim\varphi x$	(3)
(3), *2.03	⊦ $\forall x \sim\varphi x \rightarrow \sim \exists x\, \varphi x$	(4)
(2), (4), *3.2, *4.01	⊦ $\sim\exists x\, \varphi x \rightarrow \forall x \sim\varphi x$	

or:

*4.13	⊦ $\forall x \sim\varphi x \leftrightarrow \sim\sim \forall x \sim\varphi x$	(1)
(1), *4.21	⊦ $\sim\sim\forall x \sim\varphi x \leftrightarrow \forall x \sim\varphi x$	(2)
(2), *10.01	⊦ $\sim\exists x\, \varphi x \leftrightarrow \forall x \sim\varphi x$	

*10.253 ⊦ $\sim\forall x\, \varphi x \leftrightarrow \exists x \sim\varphi x$

Proof:

*10.1	⊦ $\forall x\, \varphi x \rightarrow \varphi y$	(1)
*2.12	⊦ … $\rightarrow \sim\sim\varphi y$	(2)
(2), *9.13, *10.21	⊦ $\forall x\, \varphi x \rightarrow \forall y \sim\sim\varphi y$	(3)
(3), *2.16	⊦ $\sim\forall x \sim\sim\varphi x \rightarrow \sim\forall x\, \varphi x$	(4)
(4), *10.01	⊦ $\exists x \sim\varphi x \rightarrow \sim\forall x\, \varphi x$	(5)
*10.1	⊦ $\forall x \sim\sim\varphi x \rightarrow \sim\sim\varphi y$	(6)
by analogy, *2.14	⊦ $\sim\forall x\, \varphi x \rightarrow \exists x \sim\varphi x$	(7)
(5), (7), *3.2, *4.01	⊦ $\sim\forall x\, \varphi x \leftrightarrow \exists x \sim\varphi x$	

*10.26 $\vdash (\forall x\ (\varphi x \rightarrow \psi x) \wedge \varphi y) \rightarrow \psi y$

Proof:

*10.1	$\vdash \forall x\ (\varphi x \rightarrow \psi x) \rightarrow (\varphi y \rightarrow \psi y)$	(1)
(1), *3.31	$\vdash (\forall x\ (\varphi x \rightarrow \psi x) \wedge \varphi y) \rightarrow \psi y$	

*10.27 $\vdash \forall x\ (\varphi x \rightarrow \psi x) \rightarrow (\forall x\ \varphi x \rightarrow \forall x\ \psi x)$

Proof:

*10.14	$\vdash (\forall x\ (\varphi x \rightarrow \psi x) \wedge \forall x\ \varphi x) \rightarrow ((\varphi y \rightarrow \psi y) \wedge \varphi y)$	(1)
*3.35	$\vdash \ldots \rightarrow \psi y$	(2)
(2), *9.13, *10.21	$\vdash (\forall x\ (\varphi x \rightarrow \psi x) \wedge \forall x\ \varphi x) \rightarrow \forall y\ \psi y$	(3)
(3), *3.3	$\vdash \forall x\ (\varphi x \rightarrow \psi x) \rightarrow (\forall x\ \varphi x \rightarrow \forall x\ \psi x)$	

*10.271 $\vdash \forall x\ (\varphi x \leftrightarrow \psi x) \rightarrow (\forall x\ \varphi x \leftrightarrow \forall x\ \psi x)$

Proof:

*2.08, *4.01	$\vdash \forall x\ (\varphi x \leftrightarrow \psi x) \rightarrow \forall x\ ((\varphi x \rightarrow \psi x) \wedge (\psi x \rightarrow \varphi x))$	(1)
*10.22	$\vdash \ldots \rightarrow (\forall x\ (\varphi x \rightarrow \psi x) \wedge \forall x\ (\psi x \rightarrow \varphi x))$	(2)
*3.26	$\vdash \ldots \rightarrow \forall x\ (\varphi x \rightarrow \psi x)$	(3)
*10.27	$\vdash \ldots \rightarrow (\forall x\ \varphi x \rightarrow \forall x\ \psi x)$	(4)
by analogy, *3.27	$\vdash \forall x\ (\varphi x \leftrightarrow \psi x) \rightarrow (\forall x\ \psi x \rightarrow \forall x\ \varphi x)$	(5)
(4), (5), *3.43, *4.01	$\vdash \forall x\ (\varphi x \leftrightarrow \psi x) \rightarrow (\forall x\ \varphi x \leftrightarrow \forall x\ \psi x)$	

*10.28 $\vdash \forall x\ (\varphi x \rightarrow \psi x) \rightarrow (\exists x\ \varphi x \rightarrow \exists x\ \psi x)$

Proof:

*10.1	$\vdash \forall x\ (\varphi x \rightarrow \psi x) \rightarrow (\varphi y \rightarrow \psi y)$	(1)
*2.16	$\vdash \ldots \rightarrow (\sim\psi y \rightarrow \sim\varphi y)$	(2)
(2), *9.13, *10.21	$\vdash \forall x\ (\varphi x \rightarrow \psi x) \rightarrow \forall y\ (\sim\psi y \rightarrow \sim\varphi y)$	(3)
*10.27	$\vdash \ldots \rightarrow (\forall x\ \sim\psi x \rightarrow \forall x\ \sim\varphi x)$	(4)
*2.16, *10.01	$\vdash \ldots \rightarrow (\exists x\ \varphi x \rightarrow \exists x\ \psi x)$	

*10.3 $\vdash (\forall x\ (\varphi x \rightarrow \psi x) \wedge \forall x\ (\psi x \rightarrow \chi x)) \rightarrow \forall x\ (\varphi x \rightarrow \chi x)$

Proof:

*10.22	$\vdash (\forall x\ (\varphi x \rightarrow \psi x) \wedge \forall x\ (\psi x \rightarrow \chi x))$	
	$\leftrightarrow \forall x\ ((\varphi x \rightarrow \psi x) \wedge (\psi x \rightarrow \chi x))$	(1)

(1), *4.01, *3.26	$\vdash (\forall x\ (\varphi x \to \psi x) \wedge \forall x\ (\psi x \to \chi x))$	
	$\to \forall x\ ((\varphi x \to \psi x) \wedge (\psi x \to \chi x))$	(2)
*3.33, *9.13	$\vdash \forall x\ (((\varphi x \to \psi x) \wedge (\psi x \to \chi x)) \to (\varphi x \to \chi x))$	(3)
(3), *10.27	$\vdash \forall x\ ((\varphi x \to \psi x) \wedge (\psi x \to \chi x)) \to \forall x\ (\varphi x \to \chi x)$	(4)
(2), (4), *2.06	$\vdash (\forall x\ (\varphi x \to \psi x) \wedge \forall x\ (\psi x \to \chi x)) \to \forall x\ (\varphi x \to \chi x)$	

or:

*10.14	$\vdash (\forall x\ (\varphi x \to \psi x) \wedge \forall x\ (\psi x \to \chi x)) \to ((\varphi y \to \psi y) \wedge (\psi y \to \chi y))$	(1)
*3.33	$\vdash \ldots \to (\varphi y \to \chi y)$	(2)
(2), *9.13, *10.21	$\vdash (\forall x\ (\varphi x \to \psi x) \wedge \forall x\ (\psi x \to \chi x)) \to \forall x\ (\varphi x \to \chi x)$	

*11.2 $\vdash \forall x\ \forall y\ \varphi xy \leftrightarrow \forall y\ \forall x\ \varphi xy$

Proof:

*10.1	$\vdash \forall x\ \forall y\ \varphi xy \to \forall y\ \varphi zy$	(1)
*10.1	$\vdash \ldots \to \varphi zw$	(2)
(2), *9.13, *10.21	$\vdash \forall x\ \forall y\ \varphi xy \to \forall z\ \varphi zw$	(3)
(3), *9.13, *10.21	$\vdash \forall x\ \forall y\ \varphi xy \to \forall w\ \forall z\ \varphi zw$	(4)
by analogy	$\vdash \forall y\ \forall x\ \varphi xy \to \forall z\ \forall w\ \varphi zw$	(5)
(4), (5), *3.2, *4.01	$\vdash \forall x\ \forall y\ \varphi xy \leftrightarrow \forall y\ \forall x\ \varphi xy$	

*11.26 $\vdash \exists x\ \forall y\ \varphi xy \to \forall y\ \exists x\ \varphi xy$

Proof:

*10.1	$\vdash \forall y\ \varphi xy \to \varphi xz$	(1)
(1), *9.13	$\vdash \forall x\ (\forall y\ \varphi xy \to \varphi xz)$	(2)
(2), *10.28	$\vdash \exists x\ \forall y\ \varphi xy \to \exists x\ \varphi xz$	(3)
(3), *9.13, *10.21	$\vdash \exists x\ \forall y\ \varphi xy \to \forall y\ \exists x\ \varphi xy$	

*11.51 $\vdash \exists x\ \forall y\ \varphi xy \leftrightarrow {\sim}\forall x\ \exists y\ {\sim}\varphi xy$

Proof:

*10.253	$\vdash {\sim}\forall y\ \varphi xy \leftrightarrow \exists y\ {\sim}\varphi xy$	(1)
(1), *9.13	$\vdash \forall x\ ({\sim}\forall y\ \varphi xy \leftrightarrow \exists y\ {\sim}\varphi xy)$	(2)
(2), *10.271	$\vdash \forall x\ {\sim}\forall y\ \varphi xy \leftrightarrow \forall x\ \exists y\ {\sim}\varphi xy$	(3)

(3), *4.11	⊢ ~∀x ~∀y φxy ↔ ~∀x ∃y ~φxy	(4)
(4), *10.01	⊢ ∃x ∀y φxy ↔ ~∀x ∃y ~φxy	

… etc.

Deductions with *9.14 and *9.15

*10.14 ⊢ (∀x φx ∧ ∀x ψx) → (φy ∧ ψy)

Proof:

*10.1	⊢ ∀x φx → φy	(1)
*10.1 [φx ≔ ψx]	⊢ ∀x ψx → ψy	(2)
*9.14	φa is a proposition	(3)
	φy and φy take arguments of the same type	(4)
(4), *9.14	ψa is a proposition	(5)
(3), (5), primitive ideas of negation, disjunction and general quantifier, *1.01, *3.01		
	(∀x φx → φa) → ((∀x ψx → ψa) → ((∀x φx → φa) ∧ (∀x ψx → ψa))) is a proposition	(6)
(6), *9.15	(∀x φx → φy) → ((∀x ψx → ψy) → ((∀x φx → φy) ∧ (∀x ψx → ψy))) is a propositional function	(7)
(1), (2), (7), *3.2	⊢ (∀x φx → φy) ∧ (∀x ψx → ψy)	(8)
(8), *3.47	⊢ (∀x φx ∧ ∀x ψx) → (φy ∧ ψy)	

where (4) was a tacit presupposition.

... etc.

Primitive idea

Predicative[1] function (φ!x)

[1] Elementary propositional function of x

Reducibility

*12.1 $\vdash \exists\varphi\, \forall x\, (\psi x \leftrightarrow \varphi!x)$ (*Axiom of reducibility*)

Definition of identity

*13.01 $x = y$ is an abbreviation for $\forall\varphi\, (\varphi!x \rightarrow \varphi!y)$

Deduced propositions – identity

*13.1 $\vdash x = y \leftrightarrow \forall\varphi\, (\varphi!x \rightarrow \varphi!y)$

Proof:

*4.2 $\vdash \forall\varphi\, (\varphi!x \rightarrow \varphi!y) \leftrightarrow \forall\varphi\, (\varphi!x \rightarrow \varphi!y)$ (1)

(1), *13.01 $\vdash (x = y) \leftrightarrow \forall\varphi\, (\varphi!x \rightarrow \varphi!y)$

*13.101 $\vdash x = y \rightarrow (\psi x \rightarrow \psi y)$

*Proof uses *12.1*

*13.15 $\vdash x = x$ (*Law of identity)*

Proof:

*2.08 $\vdash \varphi!x \rightarrow \varphi!x$ (1)

(1), *9.13 $\vdash \forall\varphi\, (\varphi!x \rightarrow \varphi!x)$ (2)

(2), *13.1 $\vdash x = x$

*13.16 $\vdash x = y \leftrightarrow y = x$

*13.17 $\vdash (x = y \wedge y = z) \rightarrow x = z$